Dynamic Programming

Edit distance and its variants

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Lecture 17

Some slides created by or adapted from Dr. Kevin Wayne. For more information see `http://www.cs.princeton.edu/~wayne/kleinberg-tardos`. Some code reused from Python Algorithms by Magnus Lie Hetland.

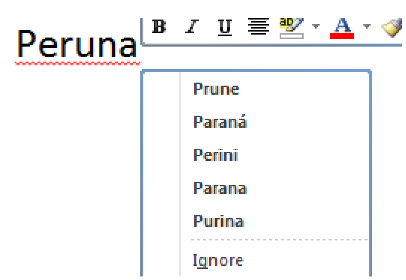
Edit distance

- Misspellings make `approximate pattern matching` an important problem
- If we are to deal with inexact string matching, we must first define a cost function telling us how far apart two strings are, i.e., a distance measure between pairs of strings.
- The edit distance is the minimum number of changes required to convert one string into another

String edit operations

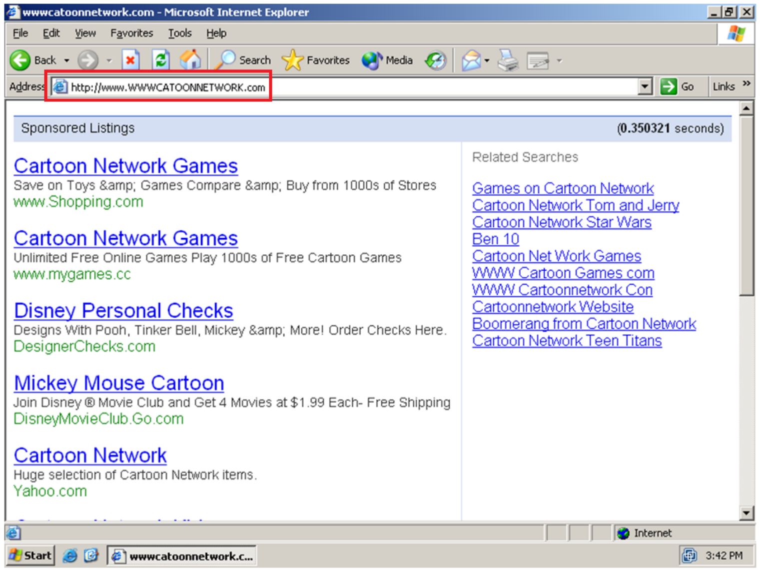
- We consider three types of changes to compute edit distance:
  1. Substitution: Change a single character from pattern s to a different character in text t, such as changing “shot” to “spot”
  2. Insertion: Insert a single character into pattern s to help it match text t, such as changing “ago” to “agog”.
  3. Deletion: Delete a single character from pattern s to help it match text t, such as changing “hour” to “our”
- This definition of edit distance is also called Levenshtein distance
- Can you think of any other natural changes that might capture a single misspelling?

Edit distance application #1



- Spell checkers identify words in a dictionary with close edit distance to the misspelled word
- But how do they order the list of suggestions?

Edit distance application #2



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Edit distance: recursive algorithm design

- Match: no substitutions

$$\overbrace{\text{shoe}}^{s_{i-1}}\ \text{s}$$
$$\underbrace{\text{show}}_{t_{j-1}}\ \underbrace{\text{s}}_{0}$$
$$(d(s_{i-1}, t_{j-1}) = 1) + 0$$
$$d(s_i, t_j) = 1$$

- Match: substitution

$$\overbrace{\text{shoe}}^{s_{i-1}}\ \text{s}$$
$$\underbrace{\text{show}}_{t_{j-1}}\ \underbrace{\text{n}}_{1}$$
$$(d(s_{i-1}, t_{j-1}) = 1) + 1$$
$$d(s_i, t_j) = 2$$

- Insertion

$$\overbrace{\text{show}}^{s_i}$$
$$\underbrace{\text{show}}_{t_{j-1}}\ \underbrace{\text{n}}_{1}$$
$$(d(s_i, t_{j-1}) = 0) + 1$$
$$d(s_i, t_j) = 1$$

- Deletion

$$\overbrace{\text{shoo}}^{s_{i-1}}\ \text{k}$$
$$\underbrace{\text{show}}_{t_j}\ \underbrace{}_{1}$$
$$(d(s_{i-1}, t_j) = 1) + 1$$
$$d(s_i, t_j) = 2$$

Recursive edit distance code

```
def string_compare(s,t):
    #start by prepending empty character to check 1st char
    s=" "+s
    t=" "+t
    P={}
    @memo
    def edit_dist(i,j):
        if i==0: return j
        if j==0: return i
        #case 1: check for match at i and j
        if s[i]==t[j]: c_match = edit_dist(i-1,j-1)
        else: c_match = edit_dist(i-1,j-1)+1
        #case 2: there is an extra character to insert
        c_ins = edit_dist(i,j-1)+1
        #case 3: there is an extra character to remove
        c_del = edit_dist(i-1,j)+1
        return min(c_match,c_ins,c_del)
    return edit_dist(len(s)-1,len(t)-1)
```

Towards a dynamic programming alternative

- We note that there are only $|s|$ possible values for `i` and $|t|$ possible values for `j` when invoking `edit_dist(i,j)` recursively
- This means there are at most $|s| \cdot |t|$ recursive function calls to cache in an iterative version
- The table is a two-dimensional matrix `C` where each of the $|s| \cdot |t|$ cells contains the cost of the optimal solution of this subproblem
- We just need a clever way to calculate the cost for each entry based on only a small subset of already-computed values.

Evaluation order

- To determine the value of cell (i, j) we need three values to already be computed: the cells $(i-1, j-1)$, $(i, j-1)$, and $(i-1, j)$.
- Any evaluation order with this property will do, including the row-major order used in the upcoming code
- But there are plenty of other valid orderings
- Think of the cells as vertices, where there is an edge (i, j) if cell is value is needed to compute cell j. Any topological sort of this DAG provides a proper evaluation order.

Edit distance: dynamic programming code

```
def iter_string_compare_lists(s,t):
    C,s,t =[]," "+s," "+t #prepend empty character for edge case
    C.append(range(len(t)+1)) #initialize cost data structure
    for i in range(len(s)):
        C.append([i+1])
    for i in range(1,len(s)): #go through all characters of s
        for j in range(1,len(t)):
            #case 1: check for match at i and j
            if s[i]==t[j]: c_match = C[i-1][j-1]
            else: c_match = C[i-1][j-1]+1
            #case 2: there is an extra character to insert
            c_ins = C[i][j-1]+1
            #case 3: there is an extra character to remove
            c_del = C[i-1][j]+1
            c_min=min(c_match,c_ins,c_del)
            C[i].append(c_min)
    return C[i][j]
```


Edit distance: DP with cost table as dictionary

```
def iter_string_compare(s,t):
    C,s,t ={}," "+s," "+t #prepend empty character for edge case
    for j in range(len(t)): #initialize cost data structure
        C[0,j]=j
    for i in range(1,len(s)):
        C[i,0]=i
    for i in range(1,len(s)): #go through all chars of s
        for j in range(1,len(t)):
            #case 1: check for match at i and j
            if s[i]==t[j]: c_match = C[i-1,j-1]
            else: c_match = C[i-1,j-1]+1
            #case 2: there is an extra character to insert
            c_ins = C[i,j-1]+1
            #case 3: there is an extra character to remove
            c_del = C[i-1,j]+1
            c_min=min(c_match,c_ins,c_del)
            C[i,j]=c_min
    return C[i,j]
```

Building edit distance cache

s: run
t: drain

C	_	d	r	a	i	n
_	0	← 1 ← 1	← 2	← 3	← 4	← 5
r	↑ 1	↖ 1	↖ 1 ↖ 1	← 2	← 3	← 4
u	↑ 2	↖ 2	↖ 2	↖ 2 ↖ 2	← 3 ← 3	← 4
n	↑ 3	↖ 3	↖ 3	↖ 3	↖ 3	↖ 3 ↖ 3

Steps to turn "run" into "drain"

1. Insert d
2. Keep r
3. Substitute a for u
4. Insert i
5. Keep n

Edit distance exercises

- Build cost table by hand following DP algorithm
  1. s: bear, t: pea
  2. s: farm, t: for
- Performance cost of DP edit distance
  - Operations: $\Theta(|s| \cdot |t|)$
  - Storage: $\Theta(|s| \cdot |t|)$

Variation of edit distance: approximate substring matching

- Suppose we want to find the best close match to a smaller word in a larger string (e.g., find the closest match to "Tulsa" in "SMU Tulda Rice")
- We need to modify our existing code in two ways
  1. Cost table initialization: all starting costs C[0,j] should be set to 0
  2. Return the finishing cell C[i,k] that minimizes the overall cost

Substring matching code

```
def iter_substring_match(s,t):
    C,s,t ={}," "+s," "+t #prepend empty character for edge case
    for j in range(len(t)): #initialize cost data structure
        C[0,j]=0 #changed: ignore cost of preceding unmatched text
    for i in range(1,len(s)):
        C[i,0]=i
    for i in range(1,len(s)): #go through all chars of s
        for j in range(1,len(t)):
            #case 1: check for match at i and j
            if s[i]==t[j]: c_match = C[i-1,j-1]
            else: c_match = C[i-1,j-1]+1
            #case 2: there is an extra character to insert
            c_ins = C[i,j-1]+1
            #case 3: there is an extra character to remove
            c_del = C[i-1,j]+1
            c_min=min(c_match,c_ins,c_del)
            C[i,j]=c_min
    finj = min([(C[i,k],k) for k in range(1,len(t)-1)])
    return "with edit dist %i, %s morphs into %s finishing at position %
```

Excercise: substring matching cache

s: Tulsa
t: SMU Tulda Rice

C	␣	S	M	U	␣	T	u	l	d	a	␣	R	i	c	e
␣	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
T	↑1	↑1	↑1	↑1	↑1	↑0	↑1	↑1	↑1	↑1	↑1	↑1	↑1	↑1	↑1
u	↑2	↑2	↑2	↑2	↑2	↑1	↖0	←1	↑2	↑2	↑2	↑2	↑2	↑2	↑2
l	↑3	↑3	↑3	↑3	↑3	↑2	↑1	↖0	←1	←2	←3	↑3	↑3	↑3	↑3
s	↑4	↑4	↑4	↑4	↑4	↑3	↑2	↑1	↖1	←2	←3	↑4	↑4	↑4	↑4
a	↑5	↑5	↑5	↑5	↑5	↑4	↑3	↑2	↖2	↖1	←2	←3	←4	↑5	↑5

Substring ending at position 9 (“Tulda”) is the closest substring to “Tulsa”

Variation of edit distance: longest common subsequence

- We might want to find the longest scattered sequence of characters within both strings
- For example, the longest common subsequence of “republican” and “democrat” is “eca”
- To get the longest subsequence, we can still allow insertions and deletions, but substitutions are forbidden
- We can change the edit distance code to behave as before on matches where the last characters are the same, but never select a substitution